# <u>Some questions on Square and Cube Formulae</u>

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|---|---|
| $x^2 - y^2 = (x + y)(x - y)$ | $x^3 \pm y^3 = (x \pm y)(x^2 \mp xy + y^2)$ |
| $(x \pm y)^2 = x^2 \pm 2xy + y^2$ | $(x \pm y)^3 = x^3 \pm 3x^2y + 3xy^2 \pm y^3$ |

**1.** Factorize **(a)** $x^3 + 2x - 12$
**(b)** $1 + x + 4x^3$
**(c)** $x^4 + 64$

**2.** If $x^3 - y^3 = 217$, solve $x, y$ where $x, y$ are integers.

**3.** Find the value of $\sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}} + \sqrt[3]{3 - \frac{11}{3}\sqrt{\frac{2}{3}}}$ .

**Answers:**

**1.** **(a)** $x^3 + 2x - 12 = (x^3 - 8) + (2x - 4) = (x - 2)(x^2 + 2x + 4) + 2(x - 2)$

$= (x - 2)[(x^2 + 2x + 4) + 2] = \underline{\underline{(x - 2)(x^2 + 2x + 6)}}$

**(b)** $1 + x + 4x^3 = (1 + 8x^3) + (x - 4x^3) = (1 + 2x)(1 + 2x + 4x^2) + x(1 - 4x^2)$

$= (1 + 2x)[(1 + 2x + 4x^2) + x] = \underline{\underline{(1 + 2x)(1 + 3x + 4x^2)}}$

**(c)** $x^4 + 64 = [(x^2)^2 + 16x^2 + 64] - 16x^2 = (x^2 + 4)^2 - (4x)^2$

$= \underline{\underline{(x^2 + 4x + 4)(x^2 - 4x + 4)}}$

**2.** $x^3 - y^3 = 217$

$(x - y)(x^2 + xy + y^2) = 1 \times 217 = 7 \times 31$

There are several cases, here we consider only

$$\begin{cases} x - y = 7 & \text{....(1)} \\ x^2 + xy + y^2 = 31 & \text{....(2)} \end{cases}$$

The other cases have no integral solutions.

From (1), $y = x - 7$ ….(3)

(3)↓(2), $x^2 + x(x - 7) + (x - 7)^2 = 31$

$x^2 + x^2 - 7x + x^2 - 14x + 49 = 31$

$3x^2 - 21x + 18 = 0$

$$x^2 - 7x + 6 = 0$$
$$(x-1)(x-6) = 0$$

$\therefore$ $x = 1$ or $6$

When $x = 1$, from (3), $y = -6$

When $x = 6$, from (3), $y = -1$

$(x, y) = (1, -6)$ or $(6, -1)$.

**3. Method 1**

Put $x = \sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}}$, $y = \sqrt[3]{3 - \frac{11}{3}\sqrt{\frac{2}{3}}}$

$x^3 = 3 + \frac{11}{3}\sqrt{\frac{2}{3}}$, $y^3 = 3 - \frac{11}{3}\sqrt{\frac{2}{3}}$

Then $x^3 + y^3 = 6$ …. (1)

And $x^3y^3 = 3^2 - \left(\frac{11}{3}\sqrt{\frac{2}{3}}\right)^2 = 9 - \frac{121}{9}\times\frac{2}{3} = 9 - \frac{242}{27} = \frac{1}{27}$

$\therefore$ $xy = \frac{1}{3}$ …. (2)

We then like to find $t = x + y$.

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$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$

$x^3 + y^3 = (x+y)[(x^2 + 2xy + y^2) - 3xy]$

$x^3 + y^3 = (x+y)[(x+y)^2 - 3xy]$

By (1) and (2),

$6 = t\left[t^2 - 3\times\frac{1}{3}\right]$

$6 = t^3 - t$

$\therefore t^3 - t - 6 = 0$

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**Or start with**

$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$

$(x+y)^3 = x^3 + y^3 + 3xy(x+y)$

By (1) and (2),

$t^3 = 6 + 3\times\frac{1}{3}t$

$\therefore t^3 - t - 6 = 0$

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Now, $t^3 - t - 6 = 0$

$$(t^3 - 8) - (t - 2) = 0$$
$$(t-2)(t^2 + 2t + 4) - (t-2) = 0$$
$$(t-2)(t^2 + 2t + 3) = 0$$

$\therefore$ $t = 2$ ( $t^2 + 2t + 3 = 0$ has no real solution since $\Delta = 2^2 - 4(3) < 0$ )

$\therefore$ $\sqrt[3]{3 + \frac{11}{3}\sqrt{\frac{2}{3}}} + \sqrt[3]{3 - \frac{11}{3}\sqrt{\frac{2}{3}}} = 2$

**Method 2**

We would like to find a, b ( $\sqrt{b}$ is real surd) such that

$$3+\frac{11}{3}\sqrt{\frac{2}{3}}=\left(a+\sqrt{b}\right)^3=a^3+3a^2\sqrt{b}+3ab+b\sqrt{b} \quad \ldots. \quad (1)$$

$$3-\frac{11}{3}\sqrt{\frac{2}{3}}=\left(a-\sqrt{b}\right)^3=a^3-3a^2\sqrt{b}+3ab-b\sqrt{b} \quad \ldots. \quad (2)$$

Obviously, $\sqrt{b}=\sqrt{\frac{2}{3}}$, or $b=\frac{2}{3}$ is a good choice.

(1) + (2), $2a^3+6ab=6$

$$2a^3+6a\left(\frac{2}{3}\right)=6$$

$$a^3+2a-3=0$$

$$\left(a^3-1\right)+\left(2a-2\right)=0$$

$$(a-1)\left(a^2+a+1\right)+2(a-1)=0$$

$$(a-1)\left(a^2+a+3\right)=0$$

$\therefore$ $a=1$ ( $a^2+a+3=0$ has no real solution since $\Delta=1^2-4(3)<0$ )

So, $3+\frac{11}{3}\sqrt{\frac{2}{3}}=\left(1+\sqrt{\frac{2}{3}}\right)^3$, $3-\frac{11}{3}\sqrt{\frac{2}{3}}=\left(1-\sqrt{\frac{2}{3}}\right)^3$

$$\sqrt[3]{3+\frac{11}{3}\sqrt{\frac{2}{3}}}+\sqrt[3]{3-\frac{11}{3}\sqrt{\frac{2}{3}}}=\left(1+\sqrt{\frac{2}{3}}\right)+\left(1-\sqrt{\frac{2}{3}}\right)=2$$